§2.9 Dimension and Rank

Given a subspace, why do we care about a basis instead of just a spanning set? As it turns out, every element (vector) of a subspace of $\mathbb{R}^n$ can be written <u>uniquely</u> as a linear combination of basis vectors.

<u>Example</u>

Take $\mathbb{R}^3$ with basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad e_1 \quad\quad e_2 \quad\quad e_3$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot e_1 + 2 \cdot e_2 \cdot 3 e_3$$

and this expression in terms of e_1, e_2, e_3 is unique

But there are lots of choices for a basis of a given subspace!

<u>Defn</u>

Suppose H is a subspace of $\mathbb{R}^n$ and $\mathcal{B} = \{b_1, \dots, b_\ell\}$ is a basis. Then any x in H can be written as

$$x = c_1 b_1 + c_2 b_2 + \cdots + c_\ell b_\ell$$

we call $c_1, \dots, c_\ell$ the coordinates of x relative to $\mathcal{B}$. The vector of $\mathbb{R}^\ell$

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_\ell \end{bmatrix}$$

is the coordinate vector of x relative to $\mathcal{B}$.

The previous example is nice since $\mathcal{B} = \{e_1, e_2, e_3\}$ and

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

But this is not always the case! Only happens with standard basis!

Example

Let $H = \operatorname{span}\left\{ \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right\}$ with basis $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right\}$

Check this is a basis!

Let $x = \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix}$. Is x in H? If so, what is $[x]_{\mathcal{B}}$?

Solution: x is in H if we can find c_1, c_2 such that

$$c_1 \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix}$$

solving this vector equation is the same as solving the matrix equation

$$\begin{bmatrix} 3 & -1 \\ 4 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & -1 & 7 \\ 4 & 0 & 8 \\ 2 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1/3 & 7/3 \\ 0 & 4/3 & -4/3 \\ 0 & 11/3 & -11/3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

Thus $x = \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$

so $[x]_\beta = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Theorem

Given a subspace H of $\mathbb{R}^n$, every basis of H has the same number of vectors. In other words if

$$\mathcal{B}_1 = \{v_1, \dots, v_k\} \qquad \mathcal{B}_2 = \{w_1, \dots, w_\ell\}$$

are both bases for H, then $k = \ell$.

Warning!

we're not saying the vectors are equal, just that the number of vectors is.

Defn: If H is a nonzero subspace, the dimension of H is the number of vectors in any basis of H. We notate it by $\dim H$.

Remarks

- As a convention, the zero subspace has dimension zero, $\dim \{0\} = 0$.
  - This makes sense since the zero subspace has no basis! To see this, remember $\{0\}$ is linearly dependent.

    set with just the zero vector, not zero subspace.
  - In particular, if H is a subspace and $\dim H = 0$ then $H = \{0\}$
- $H = \text{span}\left\{ \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right\}$ is 2-dimensional (plane in $\mathbb{R}^3$)

Definition: Given an $m \times n$ matrix A, the rank of A is the dimension of $\text{Col}(A)$ (recall this is a subspace of $\mathbb{R}^m$!).

Column spaces and Null spaces of a matrix will be our favorite types of subspaces. Even though $\text{Col}(A) \subseteq \mathbb{R}^m$ and $\text{Null } A \subseteq \mathbb{R}^n$ (here A is $m \times n$) are subspaces of different spaces, their dimensions are related!

Example (from last time 2/15)

$$A = \begin{bmatrix} 1 & 2 & 3 & 6 & 4 \\ 0 & 1 & 4 & 0 & 2 \\ 0 & 1 & 4 & 0 & 4 \end{bmatrix}$$

a) Find $\dim \text{Nul}(A)$

b) Find $\text{rank } A$

Solution:

From last time, we know A can be row reduced to

$$A \sim \begin{bmatrix} 1 & 0 & -5 & 6 & 0 \\ 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus

a) $\left\{ \begin{bmatrix} 5 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\text{Nul}(A)$

$\Rightarrow \dim \text{Nul}(A) = \boxed{2}$

b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} \right\}$ is a basis for $\text{Col}(A)$

$\Rightarrow \text{rank } A = \dim \text{Col}(A) = \boxed{3}$

Notice that in general, for a matrix A

- $\#$ pivot columns $= \dim \text{Col}(A) = \text{rank } A$
- $\#$ non-pivot columns $= \#$ free variables $= \dim \text{Nul}(A)$

Clearly every column either has a pivot or not so

$\#$ pivot columns $+$ $\#$ non-pivot columns $=$ $\#$ columns

Thus we obtain the following result:

Theorem (Rank-Nullity) (Rank Theorem)

If A is a matrix with n columns, then

$$\text{rank } A + \dim \text{Nul}(A) = n$$

As a consequence, we obtain the next result:

Theorem (Basis Theorem)

Let H be a p-dimensional subspace of $\mathbb{R}^n$.

a) If $\{v_1, ---, v_p\}$ is a linearly independent set of p vectors in H, then $\{v_1, --, v_p\}$ is a basis of H.

b) If $\{w_1, ---, w_p\}$ is a set of p vectors in H that span all of H, then $\{w_1, --, w_p\}$ is a basis of H.

Proof

a) We just need to show $\{v_1, ---, v_p\}$ spans H. Since this set is linearly independent $Ax=0$ has only the trivial solution where $A = [v_1 | \cdots | v_p]$. Thus $\text{Nul}(A) = \{0\}$ so $\dim \text{Nul}(A) = 0$, so $\text{rank } A = p$ by rank-nullity theorem. Thus every column of A is a pivot column so $\{v_1, ---, v_p\}$ spans H.

b) We just need to show $\{w_1, \dots, w_p\}$ is linearly independent. $\{w_1, \dots, w_p\}$ spans H so every column of $A = [w_1 | \cdots | w_p]$ is a pivot column. Thus rank $A = p$, hence $\dim \text{Nul}(A) = 0$ by rank-nullity theorem. However, this means $\text{Nul}(A) = \{0\}$ so $Ax = 0$ has only the trivial solution so $\{w_1, \dots, w_p\}$ is linearly independent.

In particular, if A is an $n \times n$ square matrix, we can extend the invertible matrix theorem

Invertible matrix theorem (Cont.) (2/12, 2/15)

The following are equivalent; for A an $n \times n$ matrix

a) A is invertible

Look at your notes 2/12 ; 2/15

l)

m) The columns of A form a basis for $\mathbb{R}^n$

n) $\text{Col}(A) = \mathbb{R}^n$

o) $\text{rank}(A) = n$

p) $\dim \text{Nul}(A) = 0$

q) $\text{Nul}(A) = \{0\}$